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SIMPLICIAL APPROXIMATION OF SOLUTIONS TO THE NON-LINEAR COMPLEMENTARITY PROBLEM

by

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# SIMPLICIAL APPROXIMATION OF SOLUTIONS TO THE NONLINEAR COMPLEMENTARITY PROBLEM

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ABSTRACT

Ideas of a simplicial variable dimension restart algorithm to approximate fixed points on $R^n$ developed by the authors and the linear complementarity problem algorithm of Talman and Van der Heyden are combined to develop a simplicial variable dimension restart algorithm for the nonlinear complementarity problem with lower and upper bounds (if any) on the variables. The variable dimension feature of the algorithm is not only caused by the algorithm but also by the complementarity conditions on the variables. If not all the bounds are finite, a convergence condition is given to guarantee the finiteness of the method to find an approximate solution. Finally, two applications are discussed.

KEY WORDS: simplicial algorithm, triangulation, nonlinear complementarity, stationary point

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## 1. Introduction.

The nonlinear complementarity problem arises from many problems in the field of mathematical programming, game theory and economic equilibrium. To find a solution to a nonlinear complementarity problem (NLCP) several simplicial algorithms have been proposed. Merrill [10] converted the problem into a fixed point problem and developed the so-called artificial restart algorithm for the latter problem. This approach has been followed by many authors, see e.g. Todd [15] and Allgower and Georg [1]. Other authors have adjusted simplicial fixed point algorithms to solve the NLCP. For example, Fisher and Gould [2] modified Scarf's fixed point algorithm and Lüthi [8, 9] adapted Merrill's algorithm. Recently, Reiser [11] developed a restart algorithm for the NLCP which does not need an extra dimension and which generates a path of simplices of varying dimension.

The first approach has the disadvantage that the problem is transformed into a fixed point (or zero point) problem, so that essential information about the structure of the problem is lost. On the other hand the algorithms of Lüthi and Reiser only deal with integer labelling, which in many cases seems to be less efficient than vector labelling.

In this paper we propose a vector labelling algorithm to solve the NLCP in which we combine ideas used in the papers of Van der Laan and Talman [5] and Talman and Van der Heyden [14].
In the first paper, a class of variable dimension restart algorithms was developed, from which one of the extreme cases is very similar to Reiser's algorithm. However, this algorithm was developed both for integer and vector labelling and exploits the K' instead of the K triangulation. In [14] the above mentioned class of algorithms was adapted to solve the linear complementarity problem, allowing for an arbitrary nonnegative starting point and making use of the boundary conditions on the variables.

The algorithm we will present in this paper allows for making use of the boundary conditions of the NLCP. The general version of the latter problem is stated as follows. For given vectors a and b in $R^n$ with $a_i < b_i$ for all i, find x such that for all $i = 1,\dots,n$,

$$\begin{array}{lll} x_i = a_i & \text{implies } f_i(x) \geq 0 & \\ a_i < x_i < b_i & \text{implies } f_i(x) = 0 & \qquad (1.1) \\ x_i = b_i & \text{implies } f_i(x) \leq 0 & \end{array}$$

where f is a continuous function from $R^n$ to $R^n$. Clearly, x solves (1.1) if and only if x solves the general stationary point problem: find x such that $a \leq x \leq b$ and

$$x^T f(x) \leq y^T f(x) \qquad \text{for all } a \leq y \leq b.$$

We will denote the cubic region $\{x \in R^n | a \leq x \leq b\}$ by C. We allow components of a to be minus infinite and components of b to be plus infinite. In case all the components of a are $-\infty$ and those of b are $+\infty$, problem (1.1) simplifies to the classical zero finding problem on $R^n$. When a is equal to the zero vector and all the components of b are equal to plus infinity, problem (1.1) is the classical NLCP.

The paper is organized as follows. In the next section we describe the limiting path which will be followed by the algorithm. This path is the union of 1-manifolds in t-dimensional regions $1 \leq t \leq n$. A piece of the path in a t-dimensional region can be followed by a sequence of t-dimensional simplices of a triangulation of C and lies either in the interior of C or on the boundary of C. A lower-dimensional piece ($t < n$) appears in the interior by construction of the algorithm and on the boundary by the complementarity conditions . In section 3 the triangulation of C is given and the accuracy of an obtained approximation is discussed. The procedure to follow the limiting path by generating a path of simplices of varying dimension is given in section 4. The convergence of the algorithm (if C is unbounded) is dealt with in section 5. In this section we also interprete the algorithm as an algorithm generating a path of stationary points. Finally, in section 6 we give some examples of problems to which the algorithm can be applied.

2. The path of the algorithm.

In this section we describe the path of points which will be followed by the algorithm to find a solution to (1.1). Therefore, let $v$ be an arbitrary point of $C = \{x \in R^n | a \leq x \leq b\}$. For the moment we assume that $C$ is compact. Given $v$ we define a collection of paths and loops in $C$. A path is a one-manifold with two endpoints. A loop is a one-manifold without endpoints. Since $C$ is compact we must have that following a loop cycling occurs. The collection of paths and loops we will define contains a path such that $v$ is one of its endpoints and a solution point to problem (1.1) is its other endpoint. In section 4 we will discuss how this path can be followed by simplicial approximation in order to find an approximate solution to the problem.

In the sequel, let $I_n$ be the set of integers $\{1,\dots,n\}$ and let $I_{\pm n}$ be the set $\{-n, -n+1,\dots,-1,1,\dots,n\}$. Furthermore, let $P_n$ be the collection of subsets of $I_{\pm n}$ such that for each $S \in P_n$, not both $j$ and $-j$ belong to $S$, $j = 1,\dots,n$. Finally, for some $x \in C$, let $J_x = \{i \in I_n | x_i = a_i\}$ and $J^x = \{i \in I_n | x_i = b_i\}$.

Definition 2.1. Given some $v \in C$, for each $T \subset P_n$, $\mathring{A}(T)$ is the subset of $C$ such that for all $x \in \mathring{A}(T)$,

i) $x_j \in \{a_j, v_j, b_j\}$ when both $j$ an $-j$ are not in $T$,

ii) $v_j < x_j < b_j$ when $j \in T$,

iii) $a_j < x_j < v_j$ when $-j \in T$.

When for some $j$, $v_j = b_j$ ($v_j = a_j$), $\mathring{A}(T)$ is empty for each $T \in P_n$ such that $j \in T$ ($-j \in T$). Clearly $C$ is partitioned by the sets $\mathring{A}(T)$, $T \in P_n$. Let $A(T)$ denote the closure of $\mathring{A}(T)$. By definition, each nonempty set $A(T)$ is the union of $t$-dimensional subsets of $C$ where $t = |T|$, the number of elements of $T$. For $n = 2$, some sets $\mathring{A}(T)$ are pictured in figure 1.

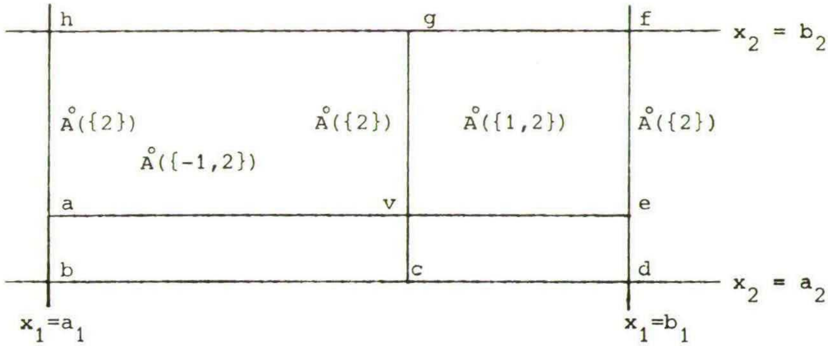


Figure 1. The subsets $\overset{\circ}{A}(T)$ for $T = \phi$, $\{2\}$, $\{1,2\}$ and $\{-1,2\}$, $n = 2$;

$\overset{\circ}{A}(\phi) = \{v, a, b, c, d, e, f, g, h\}$,

$\overset{\circ}{A}(\{2\}) = \{x \mid x_1 \in \{a_1, v_1, b_1\},\ v_2 < x_2 < b_2\}$,

$\overset{\circ}{A}(\{1,2\}) = \{x \mid v_1 < x_1 < b_1,\ v_2 < x_2 < b_2\}$,

$\overset{\circ}{A}(\{-1,2\}) = \{x \mid a_1 < x_1 < v_1,\ v_2 < x_2 < b_2\}$.

The next step is to consider subsets of C where certain conditions on f are satisfied. Therefore we first characterize the set of solution points to problem (1.1). Recall that f is a continuous function from C to $R^n$. For some x, let $\alpha(x)$ be defined by

$$\alpha(x) = \max\left[\max\{f_i(x) \mid i \notin J_x\},\ \max\{-f_i(x) \mid i \notin J^x\}\right].$$

Lemma 2.2. Let S be the set of points $\{x \in C \mid \alpha(x) \leq 0\}$. Then x is a solution to (1.1) if and only if $x \in S$.

Proof. This follows immediately from the definition of $\alpha(x)$ and the conditions of problem (1.1). □

For any $T \in P_n$, let $T^c = \{h \in I_n \mid h \text{ and } -h \text{ both not in } T\}$.

Definition 2.3. Given v, for $T \in P_n$, $\overset{\circ}{B}(T)$ is the subset of points $x \in C$ such that for some $\alpha > 0$

(i) $f_j(x) = -\alpha$ if $j \in T$

(ii) $f_j(x) = \alpha$ if $-j \in T$

(iii) $f_j(x) \quad \alpha$ when $j \in (T^c \cap J_x) \setminus J_v$

(iv) $f_j(x) < -\alpha$ when $j \in (T^c \cap J^x) \setminus J^v$

(v) $f_j(x) < \alpha$ when $j \in T^c \setminus J_x$

(vi) $f_j(x) > -\alpha$ when $j \in T^c \setminus J^x$ .

When $T \neq \emptyset$, we have $\alpha = \alpha(x)$ and $x$ is not a solution for the NLCP. In case $T = \emptyset$, we must have $\alpha > \max(0, \alpha(x))$ and the conditions (iii) - (vi) hold for a whole interval of $\alpha$'s. In particular, $v \in \mathring{B}(\emptyset)$ with $\alpha > \alpha(v)$. When for some $x \in \mathring{B}(\emptyset)$, $\alpha(x) < 0$, then $x$ is by lemma (2.2) a solution for the NLCP.

In the sequel $B(T)$ denotes the closure of $\mathring{B}(T)$. We now define the sets $\mathring{C}(T) \subset R^{2n+1}$, $T \in P_n$, by

$$\mathring{C}(T) = \{(x,y,\alpha) \in R^{2n+1} \mid y = f(x) \text{ and } \alpha > 0 \text{ such that } x \in \mathring{A}(T) \cap \mathring{B}(T)\},$$

i.e. $\mathring{C}(T)$ is the set of points $(x, f(x), \alpha) \in C \times R^n \times R_+$ such that $\alpha > 0$ and

C1. $v_j < x_j < b_j$ and $f_j(x) = -\alpha$ when $j \in T$,

C2. $a_j < x_j < v_j$ and $f_j(x) = \alpha$ when $-j \in T$,

C3. $f_j(x) > -\alpha$ and $x_j = v_j$ when $j \in T^c$ and $v_j < b_j$,

C4. $f_j(x) < \alpha$ and $x_j = v_j$ when $j \in T^c$ and $v_j > a_j$,

C5. $f_j(x) < -\alpha$ and $x_j = b_j$ when $j \in T^c$ and $v_j < b_j$,

C6. $f_j(x) > \alpha$ and $x_j = a_j$ when $j \in T^c$ and $v_j > a_j$.

Let $C(T)$ be the closure of $\mathring{C}(T)$, i.e., $C(T) = \{(x,y,\alpha) \in C \times R^n \times R_+ | y = f(x)$ and for $\alpha \quad$ , $x \in A(T) \cap B(T)\}$. For a point $(x,y,\alpha) \in C(T)$, $T \in P_n$, we say that $x_j$ is non-basic if $x_j$ equals $a_j$, $v_j$ or $b_j$, that $y_j$ is non-basic if $y_j = \alpha$ or $-\alpha$, and that $\alpha$ is non-basic if $\alpha = 0$. When non-basic a variable $x_j$, $y_j$ or $\alpha$ is said to be basic.

<u>Non-degeneracy assumption 2.4.</u> Among the $2n+1$ variables $(x,y,\alpha)$ at most $n+1$ are non-basic at any given time.

This assumption entails no loss of generality as a slight perturbation of the function can be shown to yield non-degeneracy.
Clearly, for any $(x,y,\alpha) \in \mathring{C}(T)$, $T \in P_n$, $\alpha > 0$ and, hence, $\alpha$ is basic. Furthermore $x_j$ is basic iff $y_j$ is non-basic. So, for any $(x,y,\alpha) \in \mathring{C}(T)$, there are n non-basic variables. Therefore, under some regularity conditions, the set of points of a non-empty set $\mathring{C}(T)$, $T \in P_n$, consists of a collection of 1-manifolds. Each 1-manifold either is a loop or a path. Since f is continuous and compact, each path in C(T) is bounded except for $T = \emptyset$ the path $\{(v,f(v),\alpha) | \alpha \geq \max\{0,\alpha(v)\}\}$. The latter path has one endpoint and all other paths have two endpoints. Clearly, at an endpoint there are $n+1$ non-basic variables.

We will show now that an endpoint of a path in C(T) is also an endpoint of a path in a set T' with $||T| - |T'|| = 1$ and $T \subset T'$ or $T' \subset T$, except when for such an endpoint $(x^*, f(x^*), \alpha^*)$, $\alpha^* = 0$, i.e., $x^* \in S$, the set of solution points. For $\bar{x} = (x,f(x),\alpha)$ being an endpoint of a path in C(T) with $\alpha$ positive, we must have that one of the inequalities in C1-C6 has become an equality. In case C1, $x_j$ becomes $v_j$ or $b_j$ for some $j \in T$. Then $\bar{x}$ must also be an endpoint of a path of C(T') with $T' = T\backslash\{j\}$. On this path of $\mathring{C}(T')$, $f_j(x)$ satisfies C3 or C5, i.e., at $\bar{x}$ the basic variable $x_j$ and the non-basic variable $y_j$ are exchanged. Similarly, in case C2, $\bar{x}$ belongs also to $C(T\backslash\{-j\})$. In case C3 or C5, $f_j(x)$ becomes $-\alpha$ for some $j \in T^c$. Then $\bar{x}$ is an endpoint of a path in C(T') with $T' = T \cup \{j\}$. Similarly, in case C4 or C6, $f_j(x)$ becomes $\alpha$ for some $j \in T^c$ and T' becomes $T \cup \{-j\})$. Concluding, we have that $\bar{x}$ is an endpoint of just two paths. Moreover the solution points to (1.1) induce endpoints of exactly one path.

<u>Corollary 2.5.</u> The union of all sets C(T), $T \in P_n$, consists of 1-manifolds in $C \times R^n \times R_+$. Each 1-manifold is either a loop or a path.
Each path is bounded and has exactly two endpoints except the path containing the half line $\{(v,f(v),\alpha) | \alpha \geq \max\{0,\alpha(v)\}\}$. This path has one endpoint. Each endpoint yields a solution to the NLCP.

Observe that the projection on C of the path starting with the half line $\{(v,f(v),\alpha) | \alpha \geq \max\{0,\alpha(v)\}\}$ is a union of 1-manifolds in regions A(T) for various T. Since A(T) is t-dimensional, where $t = |T|$, this projection is the union of 1-manifolds in t-dimensional regions for varying t.

Clearly, if $j \in T^c$ then either $x_j = v_j$ and we have the complementarity conditions C3 or C4 induced by the construction of the path, or $x_j \in \{a_j, b_j\}$ and we have the complementarity conditions C5 or C6 induced by the structure of the problem. In the next sections we will describe a simplicial algorithm to approximate the projection of this path on C. This will be done by following the path induced by a piecewise linear approximation to f with respect to a triangulation of C. This piecewise linear path is followed by subsequent linear programming pivot and replacement steps. The endpoint obtained in this way is an approximate solution to (1.1) and can be used as the point v for a new application of the procedure with a finer grid to improve the accuracy of the approximation. When not all the $a_i$'s or $b_i$'s are finite any path in the union of the C(T)'s can be unbounded. We return to this matter in section 5.

## 3. Triangulation and approximation.

In this section we define a triangulation of the cubic region $C = \{x \in R^n | \, a \leq x \leq b\}$ in n-dimensional simplices or n-simplices. This triangulation will be similar to the K' triangulation of $R^n$ proposed by Todd [16], see also Van der Laan and Talman [5].
Therefore, let $m_1,\ldots,m_n$ be positive integers, and define the positive vector $d = (d_1,\ldots,d_n)$ by

$$d_i = (b_i - a_i)/m_i \qquad i = 1,\ldots,n.$$

We call d the grid size vector.
Furthermore, let D be the $n \times n$ diagonal matrix with j-th diagonal element equal to $d_j$, $j = 1,\ldots,n$. Then $G^0$ will be the set of grid points of the triangulation of C defined by

$$G^0 = \{x \in C | x = a + \sum_{i=1}^{n} k_i De(i);\; k_i = 0,1,\ldots,m_i,\; i = 1,\ldots,n\}$$

where $e(i)$ is the i-th unit vector in $R^n$.
Now let $v \in G^0$ be an arbitrarily chosen gridpoint of C.
First we triangulate the n-dimensional regions $A(T)$ with $|T| = n$. Then we will show that the union of these triangulations triangulates C and that each nonempty $A(T)$, $|T| < n$, is triangulated in a similar way as $A(T)$, $|T| = n$. Observe that for $|T| = n$, $A(T)$ is a convex set.

Definition 3.1. For $T \in P_n$ with $|T| = n$ and $A(T) \neq \phi$, $G(T)$ is the collection of n-simplices $\sigma(y^1, \pi(T))$ with vertices $y^1,\ldots,y^{n+1}$ in $A(T)$ such that

(i) $y^1 = v + \sum_{h\in T} k_h De(h)$ for nonnegative integers $k_h$, $h \in T$,

(ii) $\pi(T) = (\pi_1,\ldots,\pi_n)$ is a permutation of the n elements in T,

(iii) $y^{i+1} = y^i + De(\pi_i)$ $i = 1,\ldots,n$,

where $e(h) = -e(-h)$ if $h < 0$.
Similarly as in Talman [ 13, chapter 6] it can be shown that $G(T)$ triangulates $A(T)$ and that the union G of all collections $G(T)$ of simplices fits together to a triangulation of C. In fact, this triangulation is the DK' triangulation of $R^n$ restricted to C with centre point v.
For $n = 2$, $m_1 = 6$ an $m_2 = 3$, the triangulation of C is pictured in figure 2.

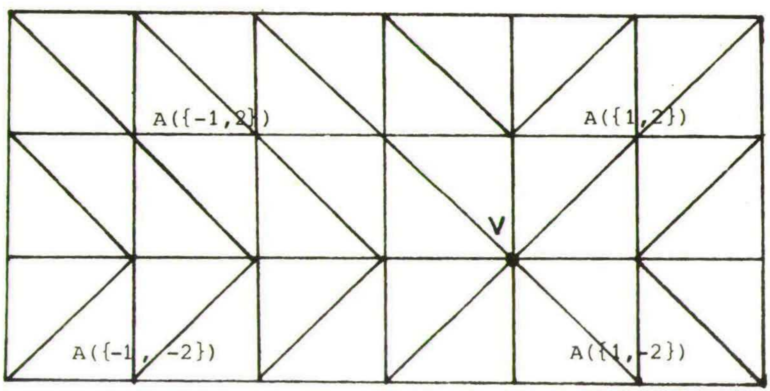


Figure 2. Triangulation of C, $n = 2$, $m_1 = 6$, $m_2 = 3$.

As shown in Spanier [12], each k-dimensional ($k < n$) subset D in region $A(T)$, such that $A(T) \cap \mathrm{aff}(D) = D$, is triangulated by $G(T)$ in k-simplices, being the intersections of simplices of $G(T)$ with D.

Lemma 3.2. Let G be the triangulation of C as defined above. Then each non-empty set $A(T)$ is triangulated by the collection $G(T)$ of t-simplices $\sigma(y^1,\pi(T))$ with vertices $y^1,\dots,y^{t+1}$ in C such that

(i) $y^1$ is a grid point in $A(T)$,

(ii) $\pi(T) = (\pi_1,\dots,\pi_t)$ is a premutation of the elements of T,

(iii) $y^{i+1} = y^i + De(\pi_i)$, $i = 1,\dots,t$ .

In fact, $A(T)$ is the union of convex subsets of C. So, with triangulating $A(T)$ we mean that each subset is triangulated (as given in lemma 3.2 ). Now, let $\bar{f}$ be the piecewise linear approximation to the function f with respect to the triangulation G, and let $\bar{\alpha}(x)$, $\mathring{B}_p(T)$, $B_p(t)$, $\mathring{C}_p(T)$ and $C_p(T)$ be defined as in section 2.1 with respect to the piecewise linear $\bar{f}$ instead of f. Without loss of generality we assume that the non-degeneracy assumption holds for any piecewise linear approximation $\bar{f}$ to f under consideration. Then $C_p(T)$ consists of a collection of piecewise linear one-manifolds each of them being either a loop or a path with two endpoints except the path having the half line $\{(v,f(v),\alpha) \mid \alpha > \max\{0,\alpha(v)\}\}$. All endpoints yield solutions to problem (1.1) with (respect to) $\bar{f}$ in stead of f. The unique path leading from the half line to a solution can be followed by linear programming steps and replacement steps as will be described in detail in the next section. The next lemma shows the accuracy of a solution to (1.1) with $\bar{f}$ instead of f.

Lemma 3.3. Let $\varepsilon,\delta > 0$ be such that $\max |x_i-y_i| < \delta$ implies $\max_i |f_i(x) - f_i(y)| < \varepsilon$ for any x and y in C. When mesh $G < \delta$ and $x^*$ is a solution to (1.1) with $\bar{f}$ instead of f, then for all i

$x_i^* = a_i$ implies $f_i(x^*) > -\varepsilon$,

$a_i < x_i^* < b_i$ implies $-\varepsilon < f_i(x^*) < \varepsilon$, and

$$x_i^* = b_i \qquad \text{implies } f_i(x^*) < \varepsilon.$$

Proof. Let $\sigma(w^1,\ldots,w^n)$ be an n-simplex of G containing $x^*$. Suppose $x_i^* = a_i$. Then $\bar{f}_i(x^*) \geq 0$, and so $f_i(x^*) \geq f_i(x^*) - \bar{f}_i(x^*) = \Sigma_j \lambda_j (f_i(x^*) - f_i(w^j)) > -\varepsilon$, where the $\lambda_i$'s are such that $x^* = \Sigma_j \lambda_j w^j$, $\Sigma_j \lambda_j = 1$ and $\lambda_j \geq 0$, $j = 1,\ldots,n$. For the case $a_i < x_i^* < b_i$ we have $\bar{f}_i(x^*) = 0$ so that

$$|f_i(x^*)| = \sum_{j=1}^{n} |\lambda_j f_i(x^*) - \lambda_j f_i(y^j)| \leq \sum_{j=1}^{n} \lambda_j |f_i(x^*) - f_i(y^i)| < \varepsilon.$$

Finally, when $x_i^* = b_i$, then $\bar{f}_i(x^*) \leq 0$. Hence,

$$f_i(x^*) \leq f_i(x^*) - \bar{f}_i(x^*) = \sum_{j=1}^{n} \lambda_j (f_i(x^*) - f_i(w^j)) < \varepsilon,$$

which proves the lemma. □

The lemma shows that a solution to (1.1) with $\bar{f}$ instead of f is an approximate solution to (1.1). For any $\varepsilon > 0$, we can find a $\delta > 0$ such that mesh $G < \delta$ gives a solution to (1.1) with $\bar{f}$ instead of f which has an inaccuracy of $\varepsilon$. Therefore taking a sequence of triangulations $G_1,\ldots G_k,\ldots,$ with mesh $G_k \to 0$ each convergent subsequence of solutions to (1.1) with the piecewise linear approximation to f with respect to $G_1,\ldots,G_k,\ldots$ instead of f converges to a solution to (1.1). As grid point v in a triangulation $G_k$ one could take the grid point closest to a solution induced by $G_{k-1}$. For $k = 1$, any grid point can be chosen as the point v. In case not all the $a_i$ or $b_i$'s are finite the results above can be easily modified.

4. The path following procedure.

To generate the piecewise linear path in C from the starting point v to a solution of (1.1) with $\bar{f}$ instead of f, each grid point of C is labelled by the vector label $\ell(x) \in R^{n+1}$ defined by

$$\ell_j(x) = f_j(x) \qquad j = 1,\ldots,n$$

$$\ell_{n+1}(x) = 1.$$

For some t-simplex $\sigma$ we define $J_\sigma = \{i \in I_n | x_i = a_i$ for all $x \in \text{int } \sigma\}$ and $J^\sigma = \{i \in I_n | x_i = b_i$ for all $x \in \text{int } \sigma\}$. Hence $J_\sigma$ ($J^\sigma$) is the set of indices i such that for all $x \in \sigma$, $x_i$ is on its lower (upper) bound. Observe that $J_\sigma \cap J^\sigma = \phi$. Now, we want to use ideas of the algorithm developed in [5] on $R^n$.

To describe their algorithm, Van der Laan and Talman [5] used a system of linear equation of order $2n\times(2n+1)$, the algorithm moving from basic solution to basic solution. Todd [17] showed that a system of $(n+1) \times (n+2)$ is sufficient, see also Kojima and Yamamoto [4]. Here we modify Todd's system to describe the algorithm on the cubic region C.

Definition 4.1. For some $T \in P_n$, a k-simplex $\sigma(w^1,\ldots,w^{k+1})$ with $k = t$ or $t-1$, $t = |T|$, is T-complete if the system of n+1 linear equations

$$\sum_{i=1}^{k+1} \lambda_i \ell(w^i) + \sum_{h\in T^c} \mu_h e(h) + \beta s(T) = e(n+1) \tag{4.1}$$

where $s(T) = (s^T,1)^T \in R^{n+1}$ and $s \in R^n$ a sign vector with $s_j = 1$ when $j \in T$, $s_j = -1$ when $-j \in T$ and $s_j = 0$ otherwise, has a solution $\lambda_1^*,\ldots,\lambda_k^*$, $\mu_h^*$, $h \in T^c$, and $\beta^*$ such that

(i) $\lambda_1^*,\ldots,\lambda_k^*, \beta^* \geq 0$,

(iia) $\mu_h^* \geq -\beta^*$ when $v_h > a_h$ or $h \notin J_\sigma$,

(iib) $\mu_h^* \leq \beta^*$ when $v_h < b_h$ or $h \notin J^\sigma$,

(iic) $\mu_h^* \leq -\beta^*$ when $v_h > a_h$ and $h \in J_\sigma$,

(iid) $\mu_h^* \geq \beta^*$ when $v_h < b_h$ and $h \in J^\sigma$.

Theorem 4.2. For some $T \in P_n$, let $\sigma(w^1,\ldots,w^{k+1})$, $k = t$ or $t-1$, be a T-complete k-simplex with solution $(\lambda^*, \mu^*, \beta^*)$. Then the point $x^* = \Sigma_i \lambda_i^* w^i / \Sigma_i \lambda_i^*$ lies in $B_p(T)$ with $\alpha = \bar{\alpha}^* = \beta^*/\Sigma_i \lambda_i^*$.

Proof. Because of the nondegeneracy assumption, $x^*$ is in $B_p(T)$ if according to definition 2.3 for some $\bar{\alpha}^* \geq 0$

(i) $\bar{f}_j(x^*) = -\bar{\alpha}^*$ when $j \in T$,

(ii) $\bar{f}_j(x^*) = \alpha^*$ when $-j \in T$,

(iii) $\bar{f}_j(x^*) \geq \bar{\alpha}^*$ when $j \in (T^C \cap J_{x^*}) \backslash J_v$,

(iv) $\bar{f}_j(x^*) \leq -\bar{\alpha}^*$ when $j \in (T^C \cap J^{x^*}) \backslash J^v$,

(v) $\bar{f}_j(x^*) \leq \bar{\alpha}^*$ when $j \in T^C \backslash J_{x^*}$, and

(vi) $\bar{f}_j(x^*) \geq -\bar{\alpha}^*$ when $j \in T^C \backslash J^{x^*}$.

For $j \notin T^C$, the j-th equation of (4.1) yields

$$\Sigma_i \lambda_i^* f_j(w^i) + \beta^* s_j = 0,$$

i.e., $\bar{f}_j(x^*) = -\beta^* s_j / \Sigma_i \lambda_i^*$ . (4.2)

Furthermore when $h \in T^C$, the h-th equation of (4.1) equals

$$\Sigma_i \lambda_i^* f_h(w^i) + \mu_h^* = 0. \tag{4.3}$$

Hence, from (iia) we have

$$\bar{f}_h(x^*) = -\mu_h^*/\Sigma\lambda_i^* \leq \beta^*/\Sigma_i\lambda_i^* \text{ when } h \notin J_{x^*} \tag{4.4}$$

and similarly from (iib)

$$\bar{f}_h(x^*) = -\mu_h^*/\Sigma\lambda_i^* \geq -\beta^*/\Sigma_i\lambda_i^* \text{ when } h \notin J^{x^*} . \tag{4.5}$$

Since $\bar{\alpha}(x)$ is defined by

$$\bar{\alpha}(x) = \max\, [\max \{\bar{f}_i(x) \,|\, i \notin J_x\},\ \max \{-\bar{f}_i(x) \,|\, i \notin J^x\}] ,$$

if follows from (4.2), (4.4) and (4.5) that $\bar{\alpha}(x^*) = \beta^*/\Sigma_i\lambda_i^*$, when $T \neq \phi$. Hence, (4.2) implies (i) and (ii) with $\bar{\alpha}^* = \beta^*/\Sigma_i\lambda_i^*$. For the case $h \in (T^C \cap J_{x^*}) \backslash J_v$, (4.3) and (iic) yield condition (iii) since

$$\bar{f}_h(x^*) = -\mu_h^*/\Sigma\lambda_i^* \geq \beta^*/\Sigma\lambda_i^* = \bar{\alpha}^*.$$

Similarly, (iv), (v) and (vi) are satisfied so that $x^* \in B_p(T)$ with $\alpha = \bar{\alpha}^*$. □

Corollary 4.3. Let $(\lambda^*, \mu^*, \beta^*)$ be a solution of a T-complete simplex $\sigma(w^1,\dots,w^{k+1})$ in A(T). Then the point $(x^*, y^*, \alpha^*)$ is in $C_p(T)$ where

$$x^* = \Sigma_i \lambda_i^* w^i / \Sigma_i \lambda_i^*,$$

$$y_j^* = -\beta^* s_j / \Sigma_i \lambda_i^* = \bar{f}_j(x^*) \quad \text{when } j \notin T^c,$$

$$y_h^* = -\mu_h^* / \Sigma_i \lambda_i^* = \bar{f}_h(x^*) \quad \text{when } h \in T^c,$$

and

$$\alpha^* = \beta^* / \Sigma_i \lambda_i^* .$$

Now, let $\sigma(w^1,\dots,w^{t+1})$ be a T-complete t-simplex, $t = |T|$. Because of the nondegeneracy assumption, $\sigma$ has just two solutions $\lambda_1,\dots,\lambda_{t+1}$, $\beta$, $\mu_h$, $h \in T^c$, with exactly one of the bounds in (i), (iia)-(iid) binding. Such a solution is called a basic solution. Since $\bar{f}$ is linear on $\sigma$, each convex combination of the two basic solution is also a solution with no bounds binding, i.e. the set of solutions yields a line segment in $\sigma$ of points in $B_p(T)$. The endpoints of the line segment correspond to the basic solutions.

Definition 4.4. A T-complete t-simplex $\sigma(w^1,\dots,w^{t+1})$ is a complete simplex when it has a basic solution $(\lambda^*, \mu^*, \beta^*)$ with $\beta^* = 0$.

Theorem 4.5. Let $\sigma(w^1,\dots,w^{t+1})$ be a complete simplex with solution $(\lambda^*, \mu^*, 0)$. Then the point $x^* = \Sigma_i \lambda_i^* w^i$ lies in $\sigma$ and is a solution to (1.1) with $\bar{f}$ instead of f.

Proof. Firstly, observe that from the last equation it follows immediately that $\Sigma_i \lambda_i^* = 1$. Hence $x^*$ is a convex combination of the vertices of $\sigma$. Secondly, suppose $h \in T^c \setminus (J_\sigma \cup J^\sigma)$.
Then, by the nondegeneracy assumption, it follows from (iia) and (iib) that both $\mu_h^* > -\beta^*$ and $\mu_h^* < \beta^*$ which gives a contradiction of the fact that $\beta^* = 0$.
So, for each $h \in I_n$ we have one of the three following cases,
a) h or -h is an element of T, b) $h \in T^c \cap J_\sigma$, c) $h \in T^c \cap J^\sigma$.
In case a) the h-th equation of (4.1) equals

$$\sum_{i=1}^{t+1} \lambda_i^* f_h(w^i) = 0,$$

i.e., $\bar{f}_h(x^*) = 0$. In case b) we must have $x^*_h = a_h$, while from (iib) it follows that

$$\bar{f}_h(x^*) = \sum_{i=1}^{t+1} \lambda_i f_h(w^i) = -\mu^*_h > -\beta^* = 0.$$

Similarly, in case c) we have $x^*_h = b_h$ and $\bar{f}_h(x^*) < 0$. Hence $x^*$ solves (1.1) for $\bar{f}$. □

Corollary 4.6. Let $\sigma$ be a T complete t-simplex which is also complete. Then $t = n-s$ where $s = |J_\sigma \cup J^\sigma|$, i.e. the dimension of $\sigma$ is n minus the number of indices h such that $x_h = a_h$ or $b_h$, $x \in \operatorname{int} \sigma$.

Notice that for a complete simplex with solution $(\lambda^*, \mu^*, 0)$ (iia) - (iid) become $\mu^*_h \geq 0$ when $h \in J^\sigma$ and $\mu^*_h \leq 0$ when $h \in J_\sigma$, independent of the location of the starting point v. Therefore, when the nondegeneracy assumption holds, a t-simplex $\sigma(w^1,\ldots,w^{t+1})$ is complete iff for some index set $T \in P^n$ the system of linear equations

$$\sum_{i=1}^{t+1} \lambda_i \ell(w^i) + \sum_{h \in T^c} \mu_h e(h) = e(n+1) \tag{4.6}$$

has a solution $(\lambda^*, \mu^*)$ with $\lambda^*_i > 0$, $i = 1,\ldots,t+1$, $\mu^*_h > 0$, $h \in J^\sigma$ and $\mu^*_h < 0$, $h \in J_\sigma$. Recall that $T^c = J^\sigma \cap J_\sigma$. Moreover, it can easily be shown that when $x^*$ is a solution to (1.1) for $\bar{f}$, the simplex $\sigma(w^1,\ldots w^{t+1})$ containing $x^*$ in its interior is complete with $T^c = J^\sigma \cup J_\sigma$.

We will now consider T-complete t-simplices $\sigma$ in A(T). As shown above a T-complete t-simplex has exactly two basic solutions. Let $x^1$ and $x^2$ be the corresponding points in $\sigma$. Then the linesegment $[x^1, x^2]$ lies in $B_p(T)$ and induces a linesegment in $C_p(T)$. We will discuss how such a linesegment can be followed. Suppose first that for each of the basic solutions we have that one of the $\lambda_i$'s is zero, say $\lambda_{i_1}$ and $\lambda_{i_2}$ respectively, i.e. the two (t-1)-facets opposits $w^{i_1}$ and $w^{i_2}$ are T-complete with solution points $x^1$ and $x^2$ respectively. Then moving from $x^1$ to $x^2$ is nothing more than making a linear programming pivot step by bringing $\ell(w^{i_2})$ into the system

$$\sum_{i \neq i_2} \lambda_i \ell(w^i) + \sum_{h \in T^c} \mu_h e(h) + \beta s(T) = e(n+1).$$

So, a 1-manifold of $C_p(T)$ can be followed by generating a sequence of adjacent t-simplices in $A(T)$ with T-complete common facets by alternating replacement and pivot steps. When by a pivot step $\lambda_{i_1}$ becomes zero, the vertex $w^{i_1}$ of $\sigma(w^1, \pi(T))$ is replaced by a new vertex in a simplex $\sigma'$ sharing the facet $\tau(w^1,\ldots,w^{i_1-1}, w^{i_1+1},\ldots,w^{t+1})$ with $\sigma$.
By construction, this replacement step is unique (see section 3). A 1-manifold in $C_p(T)$ terminates as soon as either a T-complete facet lies in bd $A(T)$, or by a pivot step a basic solution is reached for which (iia)-(iid) holds with exactly one equality or $\beta^* = 0$. According to theorem 4.5, when $\beta^* = 0$, the basic solution yields an approximate solution to (1.1) (see also lemma 3.3). Now suppose that at a basic solution $(\lambda^*, \mu^*, \beta^*)$ one of the constraints of (iia)-(iid) is binding, then for certain $k \in I\underline{+}n$ with $|k| \in T^c$, $\sigma$ is also $(T \cup \{k\})$-complete. The point $x^* = \Sigma_{i=1}^{t+1} \lambda_i^* \bar{w}^i / \Sigma_i \lambda_i^*$ is then not only endpoint of a path in $C_p(T)$ but also endpoint of a path in $C_p(T \cup \{k\})$. To follow the latter path, the current system (4.1) is adapted by replacing $e(|k|)$ and $s(T)$ by $s(T \cup \{k\})$ and a pivot step is made with $\ell(\bar{w})$, where $\bar{w}$ is the vertex of the (t+1)-simplex (if any) in $A(T \cup \{k\})$ having $\sigma$ as facet opposite $\bar{w}$.

Lemma 4.7. Let $\sigma(w^1, \pi(T))$ be a $(T \cup \{k\})$-complete t-simplex in $A(T)$ for some $k \in I\underline{+}n$, $|k| \in T^c$. Then there exists exactly one simplex $\tau$ in $A(T \cup \{k\})$ having $\sigma$ as facet. Moreover,

a) $\tau = \tau(w^1, (\pi(T), k))$ when $w^1_{|k|} = v_{|k|}$

and

b) $\tau = \tau(w^1 - De(k), (k, \pi(T)))$ when $w^1_{|k|} \neq v_{|k|}$.

Proof. Since $|k| \in T^c$, it follows from the definition of $C_p(T)$, that $w^1_{|k|} = x^*_{|k|} \in \{a_{|k|}, v_{|k|}\}$ if $k < 0$, and that $w^1_k = x^*_k \in \{v_k, b_k\}$

if $k > 0$, where $x^*$ is the endpoint of the path of $C_p(T)$ in $\sigma$. Suppose, that in the cases (iia) or (iic) $\mu^*_{|k|} = -\beta^*$. Then $k < 0$ and $v_{|k|} > a_{|k|}$. Hence, $\mathring{A}(T \cup \{k\}) \neq \phi$ and $\tau$ as defined in the lemma exists and is by construction of G the unique simplex in $G(T \cup \{k\})$ having $\sigma$ as facet. Similarly, in the cases (iib) and (iid), k must be positive and $v_k < b_k$ implying that $\mathring{A}(T \cup \{k\}) \neq \phi$. □

Finally, we consider the case that a T-complete facet $\tau$ of $\sigma$ lies in bd $A(T)$. Then, there is a unique $k \in T$ such that $\tau$ is a $(t-1)$-simplex in $A(T\setminus\{k\})$. To follow the path in $C_p(T\setminus\{k\})$ having $x^* = \Sigma_{i=1}^{t+1} \lambda_i^* w^i$ as endpoint, the vector $s(T)$ is replaced by $s(T|\{k\})$ and $e(|k|)$ is reintroduced in the current system (4.1). The following lemma states how the facet $\tau$ is obtained from $\sigma$.

Lemma 4.8. Let $\sigma(w^1, \pi(T))$ be a t-simplex in $A(T)$ having a T-complete facet $\tau$ in $A(T\setminus\{k\})$ for some $k \in T$. Then,

a) $\pi_t = k$ and $\tau = \tau(w^1, (\pi_1,\ldots,\pi_{t-1}))$ if $w^1_{|k|} = v_{|k|}$

and

b) $\pi_1 = k$ and $\tau = \tau(w^1 + De(k), (\pi_2,\ldots,\pi_t))$ if $w^1_{|k|} \neq v_{|k|}$.

Proof. In case a), suppose $\pi_t \neq k$. Since $k \in T$, we must have $\pi_i = k$ for some $i < t$. This implies that $w^{i+1},\ldots,w^{t+1}$ are not in $A(T\setminus\{k\})$, which contradicts the fact that $\sigma$ has a facet in $A(T\setminus\{k\})$. Similarly, in case b), $\pi_1 \neq k$ implies $\pi_i = k$ for some $i > 1$ and therefore $w^1,\ldots,w^i$ are not in $A(T\setminus\{k\})$.

□

Notice that in case a) the permutation $(\pi_1,\ldots,\pi_{t-1})$ and in case b) $(\pi_2,\ldots,\pi_t)$ is a permutation of the elements of $T\setminus\{k\}$ so that $\tau = \tau(y^1, \pi(T\setminus\{k\}))$ is indeed a simplex of $G(T\setminus\{k\})$.
Combining all the cases above together describes how a path of $C_p(T)$ can be followed by alternating pivot and replacement steps and how a change from a path of $C_p(T)$ at an endpoint to an "adjacent" path in $C_p(T \cup \{k\})$ or $C_p(T\setminus\{h\})$ has to be performed. Therefore, we are ready to give the formal steps of the algorithm which follows the one-manifold in $\cup_T C_p(T)$ which starts with the half line $\{(v,f(v),\alpha)\,|\,\alpha > \max\{0,\alpha(v)\}\}$ for $T = \phi$ and terminates with an approximate solution to (1.1).

Step 0. Set $T = \phi$, $\pi(T) = \phi$, $t = 0$, $w^1 = v$, $\sigma = \sigma(w^1, \pi(T))$, $p = 1$.

Step 1. Calculate $\ell(w^p)$ and perform a linear programming pivot step by bringing $\ell(w^p)$ into the system of n+1 linear equations

$$\sum_{\substack{i=1 \\ i\neq p}}^{t+1} \lambda_i \ell(w^i) + \sum_{h\in T^c} \mu_h e(h) + \beta s(T) = e(n+1). \tag{4.6}$$

Step 2. When $\beta$ becomes zero, the algorithm terminates and $\Sigma_{i=1}^{t+1}\lambda_i w^i$ is an approximate solution to (1.1).
When $\mu_h$ becomes $\beta$ for some $h \in T^c$ with $v_h > a_h$, go to step 4. When $\mu_h$ becomes $\beta$ for some $h \in T^c$ with $v_h < b_h$, go to step 5. Otherwise, for a unique $q \neq p$, $\lambda_q$ becomes zero.

Step 3. When $q = t+1$ and $w^1_{|\pi_t|} = v^1_{|\pi_t|}$, go to step 6.
When $q = 1$ and $w^1_{\pi_1} = b_{\pi_1} - d_{\pi_1}$ if $\pi_1 > 0$ or $w^1_{-\pi_1} = a_{-\pi_1} + d_{-\pi_1}$ if $\pi_1 < 0$, go to step 7.
Otherwise, adapt $w^1$ and $\pi(T)$ according to table 1 by replacing $w^q$, and return to step 1 with $w^p$ equal to the new vertex of $\sigma(w^1, \pi(T))$.

| | $w^1$ becomes | $\pi(T) = (\pi_1,\ldots,\pi_t)$ becomes |
|---|---|---|
| $q = 1$ | $w^1 + De(\pi_1)$ | $(\pi_2,\ldots,\pi_t, \pi_1)$ |
| $1 < q < t+1$ | $w^1$ | $(\pi_1,\ldots,\pi_{q-2}, \pi_q, \pi_{q-1},\pi_{q+1},\ldots,\pi_t)$ |
| $q = t+1$ | $w^1 - De(\pi_t)$ | $(\pi_t, \pi_1,\ldots,\pi_{t-1})$ |

Table 1. $q$ is the index of the vertex of $\sigma(y^1, \pi(T))$ to be replaced.

Step 4. Adapt the current system of linear equations by introducing $s(T \cup \{-h\})$ and eliminating $e(h)$ and $s(T)$. When $w^1_h = v_h$, set $T = T \cup \{-h\}$ and $\pi(T) = (\pi(T),-h)$. When $w^1_h = a_h$, set $w^1 = w^1 - De(-h)$, $T = T \cup \{-h\}$ and $\pi(T) = (-h,\pi(T))$. Set $t = t+1$, $\sigma = \sigma(w^1,\pi(T))$, and return to step 1 with $p$ the index of the new vertex of $\sigma$.

Step 5. Adapt the current system of linear equations by introducing $s(T \cup \{h\})$ and eliminating $e(h)$ and $s(T)$.
When $w^1_h = v_h$, set $T = T \cup \{h\}$ and $\pi(T) = (\pi(T), h)$. When $w^1_h = b_h$, set $w^1 = w^1 - De(h)$, $T = T \cup \{h\}$ and $\pi(T) = (h, \pi(T))$. Set $t = t+1$, $\sigma = \sigma(w^1, \pi(T))$, and return to step 1 with $p$ the index of the new vertex of $\sigma$.

Step 6. Adapt the current system of linear equations by introducing $s(T\setminus\{k\})$ and $e(|k|)$ and eliminating $s(T)$, where $k = \pi_t$.
Set $T = T\setminus\{k\}$, $\pi(T) = (\pi_1,\ldots,\pi_{t-1})$,
$\sigma = \sigma(w^1, \pi(T))$, $t = t-1$, and perform a linear programming pivot step by decreasing $\mu_k$ from $\beta$ when $k > 0$ and increasing $\mu_k$ from $-\beta$ when $k < 0$ in the system

$$\sum_{i=1}^{t+1} \lambda_i \ell(w^i) + \sum_{h\in T^c} \mu_h e(h) + \beta s(T) = e(n+1).$$

Return to step 2.

Step 7. Adapt the current system of linear equations by introducing $s(T\setminus\{k\})$ and $e(|k|)$ and eliminating $s(T)$, where $k = \pi_1$.
Set $T = T\setminus\{k\}$, $\pi(T) = (\pi_2,\ldots,\pi_t)$, $w^1 = w^1 + De(k)$, $\sigma = \sigma(w^1, \pi(T))$, $t = t-1$, and perform a linear programming pivot step by increasing $\mu_k$ from $\beta$ when $k > 0$ and decreasing $\mu_k$ from $-\beta$ when $k < 0$ in the system

$$\sum_{i=1}^{t+1} \lambda_i \ell(w^i) + \sum_{h\in T^c} \mu_h e(h) + \beta s(T) = e(n+1).$$

Return to step 2.

Following these steps, the algorithm generates a unique path of adjacent simplices of variable dimension such that for some $T$ a generated simplex is of the form $\sigma(y^1, \pi(T))$ and lies in $A(T)$ whereas the common facets are $T$-complete. As soon as such a facet lies in $A(T\setminus\{k\})$ for some $k \in T$, the dimension of $\sigma$ is decreased by deleting $k$ from $T$. If, however, $\sigma$ is $(T \cup \{h\})$-complete for some $h \in T^c$, then the dimension of $\sigma$ is increased by adding $h$ to $T$.
Because of the nondegeneracy assumption and lemmas 4.7 and 4.8 all replacement steps and pivot steps are unique and feasible so that no simplex can be generated for a second time. Therefore, in case all the $a_i$'s and $b_i$'s are finite the algorithm must terminate within a finite number of steps in step 2 with a complete simplex yielding an approximate solution to (1.1). By restarting the algorithm in a grid point close to the approximate solution for a triangulation with a smaller grid size vector, the accuracy of the approximation can be improved.

Clearly, the algorithm makes use of the structure of the problem by allowing that a sequence of lower dimensional simplices are generated on bnd C.
In particular, corollary 4.6 states that a complete simplex is (n-s)-dimensional, where s is the number of indices h for which $x^*_h = a_h$ or $b_h$. Clearly, restarting the algorithm on bnd C, the size of the problem is in advance reduced to n-s. Of course, the size is increased as soon as one of the inequalities (iic) or (iid) becomes binding. On the other hand, the size is decreased as soon as for some i, $x_i$ becomes $a_i$ or $b_i$. So, the algorithm makes full use of the complementarity conditions of the problem. This should be very useful if we have to solve some master problem, such that for each function evaluation an NLCP must be solved. Then at each function evaluation, the solution to the NLCP of the previous evaluation could serve as the starting point. This reduces the size of the NLCP if some of the variables are on their upper or lower bound.

## 5. Convergence and interpretation.

As shown at the end of the previous section, the algorithm converges always when all the $a_i$'s and $b_i$'s are finite since in that case C is compact. If at least one of $a_i$'s is minus infinite or one of the $b_i$'s is infinite, the path of generated simplices could be infinite. Since, however, each compact subset of C is covered by a finite number of simplices of the triangulation and no simplex can be generated more than once, at least one component of an infinite path goes to infinity. The following theorem guarantees that the path of simplices will be finite. Therefore, let $I_-$ be the set of indices i such that $a_i$ is minus infinite, and let $I_+$ be the set of indices i such that $b_i$ is plus infinite.

Theorem 5.1. Suppose that for all $i \in I_-$ there exists an $\ell_i$ such that $f_i(x) < 0$ when $x_i < \ell_i$ and that for all $i \in I_+$ there exists a $u_i$ such that $f_i(x) > 0$ when $x_i > u_i$, where $\ell_i < u_i$ when $i \in I_+ \cup I_-$. Then the algorithm converges for any starting point v and grid size vector d.

Proof. Let C' be the subset of C defined by

$$C' = \{x \in R^n \; x_i \geq a_i \text{ when } i \notin I_-,\; x_i \geq \min(v_i, \ell_i) \text{ when } i \in I_-,$$
$$x_i \leq b_i \text{ when } i \notin I_+,\; x_i \leq \max(v_i, u_i) \text{ when } i \in I_+\}.$$

Observe that $v \in V'$ and that $C'$ is a compact subset. We will show that the algorithm cannot generate simplices outside $C'$. Since $v \in C'$, the starting simplex lies in $C'$. Now let $\tau(y^1,\ldots,y^t)$ be any facet of a t-simplex $\sigma(w^1, \pi(T))$ in $A(T)$ not in $C'$, where $A(T)$ is as defined before. Since $\sigma$ lies outside $C'$, there is at least one index i such that for all $x \in \tau$ either $i \in I_-$ and $x_i < \min(v_i, \ell_i)$ or $i \in I_+$ and $x_i > \max(v_i, u_i)$. Suppose that $i \in I_-$ and $x_i < \min(v_i, \ell_i)$. Since $x_i < \ell_i$, we must have $f_i(x) < 0$ whereas $x_i < v_i$ implies $-i \in T$.
Hence the system of linear equations with respect to $\tau$

$$\sum_{j=1}^{t} \lambda_j \ell(y^i) + \sum_{h \in T^c} \mu_h e(h) + \beta s(T) = e(n+1) \qquad (5.1)$$

does not have a feasible solution with all the $\lambda_j$'s positive and $\beta$ non-negative since the i-th equation is equal to

$$\sum_{j=1}^{t} \lambda_j f_i(y^i) - \beta = 0.$$

Similarly, $i \in I_+$ and $x_i > \max(v_i, u_i)$ imply $f_i(x) > 0$ and $i \in T$ sothat (5.1) does not have a feasible solution with the same properties of $\lambda_j$'s and $\beta$. Therefore $\tau(y^1,\ldots,y^t)$ is not T-complete and cannot be generated by the algorithm. Similarly, a simplex $\sigma(y^1, \pi(T))$ in $A(T)$ outside $C'$ cannot be T-complete. □

<u>Corollary 5.2.</u> Under the conditions of theorem 5.1, the system (1.1) has a solution and each solution lies in the set $C^*$ defined by

$$C^* = \{x \in R^n | x_i \geq a_i \text{ when } i \notin I_-,\; x_i > \ell_i \text{ when } i \in I_-,$$
$$x_i \leq b_i \text{ when } i \notin I_+, \text{ and } x_i < u_i \text{ when } i \in I_+\}.$$

As done by Van der Laan and Talman [17] for the case that all the $a_i$'s and $b_i$'s are not finite, the algorithm can be interpreted as generating a path of stationary points with respect to the function f restricted to C and to an expanding octahedron with v as centre point. To do so, let v be the starting point and let $\bar{f}$ be the piecewise linear approximation to f. Moreover, write $x \in C$ as $x = v + y^1 - y^2$ with $y^1$ and $y^2$ nonnegative complementary vectors, i.e. $y^1 \geq 0$, $y^2 \geq 0$ and $y_i^1 y_i^2 = 0$, $i = 1,\ldots,n$. Then consider the following stationary point problem. For $t \geq 0$, find $x(t)$ such that $x(t)^T\bar{f}(x(t)) \leq x^T\bar{f}(x(t))$ for all x such that $\Sigma(y_i^1+y_i^2) \leq t$, $y^1 \leq b-v$, $y^2 \leq v-a$, $y^1 \geq 0$, $y^2 \geq 0$ (where $x = v + y^1 - y^2$ with $y^{1T}y^2 = 0$). When $t = 0$, $x(0)$ must be the point v since in that case $y^1$ and $y^2$ are both the zero-vector. In general, $x(t)$ is a stationary point for the problem above when $x(t)$ solves the problem find $x^*$ such that

$$x^{*T}\bar{f}(x^*) \leq x^T\bar{f}(x^*)$$

for all $x \in C \cap D(t)$, where

$$D(t) = \{x \in R^n \mid \sum_{i=1}^{n} |x_i - v_i| \leq t\}.$$

Theorem 5.3. Let $\bar{x}$ solve the stationary point problem for some $t > 0$ and let $\bar{y}^1$ and $\bar{y}^2$ be such that $\bar{x} = v + \bar{y}^1 - \bar{y}^2$ with $\bar{y}^1 \geq 0$, $\bar{y}^2 \geq 0$ and $\bar{y}^{T1}\bar{y}^2 = 0$. Then $\bar{x} \in A(T) \cap B_p(T)$ where $T = \{i \mid \bar{y}_i^1 > 0\} \cup \{-j \mid \bar{y}_j^2 > 0\}$. Moreover, $\bar{x}$ solves (1.1) for $\bar{f}$ when $\bar{x} \in \text{int } D(t)$.

Proof. Consider the linear programming problem min $x^T\bar{f}(\bar{x})$ such that $x = v + y^1 - y^2$, $\Sigma(y_i^1 + y_i^2) \leq t$, $y^1 \leq b-v$ and $y^2 \leq v-a$ with $y^1$ and $y^2 \geq 0$ such that $y^{T1}y^2 = 0$. Let the vector $\mu$, the number $\beta$ and the vectors $\alpha^1$ and $\alpha^2$ be the corresponding simplex multipliers respectively. At the solution $\bar{x}$ the inequalities with respect to the following variables hold:

$$\begin{aligned}
\bar{x}_i &: \bar{f}_i(\bar{x}) = \mu_i \\
y_i^1 &: \mu_i + \beta + \alpha_i^1 \geq 0 \\
y_i^2 &: -\mu_i + \beta + \alpha_i^2 \geq 0 \\
\beta &: \Sigma(y_i^1 + y_i^2) \leq t \\
\mu_i &: \bar{x}_i = v_i + y_i^1 - y_i^2 \\
\alpha_i^1 &: y_i^1 \leq b_i - v_i \\
\alpha_i^2 &: y_i^2 \leq v_i - a_i
\end{aligned}$$

whereas all variables except $x$ and $\mu$ are nonnegative and $y^{1^T} y^2 = 0$. When an inequality is strict, the corresponding variable is zero.
Suppose that $\sum_{i=1}^{n} (y_i^1 + y_i^2) < t$. Then $\beta$ must be zero, so that if $y_i^1 > 0$ we have $\mu_i + \alpha_i^1 = 0$ with $\alpha_i^1 = 0$ when $y_i^1 < b_i - v_i$, and if $y_i^2 > 0$ we have $-\mu_i + \alpha_i^2 = 0$ with $\alpha_i^2 = 0$ when $y_i^2 < v_i - a_i$. Suppose $y_i^1 > 0$. Then $\bar{f}_i(\bar{x}) = \mu_i = 0$ when $y_i^1 < b_i - v_i$ and $\bar{f}_i(\bar{x}) = \mu_i \leq 0$ when $y_i^1 = b_i - v_i$. If $y_i^2 > 0$, then $\bar{f}_i(\bar{x}) = \mu_i = 0$ when $y_i^2 < v_i - a_i$ and $\bar{f}_i(\bar{x}) = \mu_i \geq 0$ when $y_i^2 = v_i - a_i > 0$. Observe that $b_i - v_i > 0$ when $y_i^1 > 0$, and $v_i - a_i > 0$ when $y_i^2 > 0$. Hence, since $\bar{x} = v + y^1 - y^2$, we have that $\bar{f}_i(\bar{x}) = 0$ when $a_i < \bar{x}_i < v_i$ and $v_i < \bar{x}_i < b_i$, that $\bar{f}_i(\bar{x}) \leq 0$ when $\bar{x}_i = b_i$ and that $\bar{f}_i(\bar{x}) \geq 0$ when $\bar{x}_i = a_i$. Now we consider the case that $\bar{x}_i = v_i$ i.e., $y_i^1 = y_i^2 = 0$. When $a_i < v_i < b_i$, we get $\alpha_i^1 = \alpha_i^2 = 0$ so that $\mu_i \geq 0$ and $-\mu_i \geq 0$, i.e., $\bar{f}_i(\bar{x}) = 0$. Note that this case is excluded under the nondegeneracy assumption. In case $v_i = a_i$, we get $\alpha_i^1 = 0$, so that $\mu_i \geq 0$, and, hence, $\bar{f}_i(\bar{x}) \geq 0$. For $v_i = b_i$, we have $\alpha_i^2 = 0$, and therefore $\bar{f}_i(\bar{x}) = \mu_i \leq 0$. Combining all these cases together imply that if $\bar{x} \in \operatorname{int} B(t)$, $\bar{x}$ is a solution to (1.1) with $\bar{f}$ instead of $f$. Now suppose that $\Sigma_{i=1}^{n}(y^1 + y_i^2) = t$. Without loss of generality we assume that $\beta > 0$.

In case both $y_i^1$ and $y_i^2$ are zero, i.e. $\bar{x}_i = v_i$, we have since $\bar{f}_i(\bar{x}) = \mu_i$

$$\bar{f}_i(\bar{x}) + \beta + \alpha_i^1 > 0 \text{ and } -\bar{f}_i(\bar{x}) + \beta + \alpha_i^2 > 0.$$

When $a_i < v_i < b_i$, both $\alpha_i^1$ and $\alpha_i^2$ are zero, so that $-\beta < \bar{f}_i(\bar{x}) < \beta$. If $\bar{x}_i = v_i = a_i$, then $\alpha_i^1 = 0$, i.e., $\bar{f}_i(\bar{x}) > -\beta$ whereas $\alpha_i^2 > 0$, making $-\bar{f}_i(\bar{x}) + \beta + \alpha_i^2 > 0$ redundant. Similarly, $\bar{f}_i(\bar{x}) < \beta$ when $\bar{x}_i = v_i = b_i$. Now suppose that $y_i^1 > 0$. Then $\bar{f}_i(\bar{x}) + \beta + \alpha_i^1 = 0$ with $\alpha_i^1 = 0$ when $y_i^1 < b_i - v_i$. So, $y_i^1 < b_i - v_i$ implies $\bar{f}_i(\bar{x}) = -\beta$ and $y_i^1 = b_i - v_i$ gives $\bar{f}_i(\bar{x}) < -\beta$. Similarly, when $y_i^2 > 0$, then $-\bar{f}_i(\bar{x}) + \beta + \alpha_i^2 = 0$, so that $\bar{f}_i(\bar{x}) \geq \beta$ with equality when $y_i^2 < v_i - a_i$. Since $x = v + y^1 - y^2$, all of this together implies $(\bar{x}, \bar{f}(\bar{x}), \beta)$ lies in $C_p(T)$, where $T = \{i | y_i^1 > 0\} \cup \{-j | y_j^2 > 0\}$. $\square$

Now, let us consider the points $(x(t), t)$ such that $x(t)$ solves the stationary point problem for $x \in C \cap D(t)$. As shown above $C \cap D(0) = \{v\}$, so that $(v,0)$ is such a point. By the nondegeneracy assumption we have that $\cup\, C_p(T)$ is a collection of paths and loops (corollary 2.5). Hence, by theorem 5.3, the set of solution points $(x(t), t)$ is also a disjoint union of paths and loops in $R^n \times R_+$. So, the algorithm can be interpreted as following a path of solution points $(x(t), t)$ in $R^n \times R_+$, having $(v,0)$ as its endpoint. Observe that when $(x(t^*), t^*)$ is a stationary point solution with $x(t^*) \in \text{int}\, D(t^*)$, then $(x(t^*), t)$ is a solution for all $t > t^*$. The point $x(t^*)$ is then a solution to (1.1) for $\bar{f}$. As long as $x(t) \in \text{bnd}\, (D(t) \cap C)$ and $\bar{f}(x(t)) \neq 0$, $(x(t), t)$ is such that $C \cap D(t)$ is a subset of $\bar{H}(\bar{f}(x(t)), x^T(t)\bar{f}(x(t)))$, where for some $p \in R^n \backslash \{0\}$, $c \in R$, $H(p,c) = \{x \in R^n | p^T x = c\}$ and $\bar{H}(p,c)$ is the half space above $H(p,c)$. In other words, $(x(t),t)$ is such that the set $C \cap D(t)$ is in the half space below the hyperplane through $x(t)$ with normal $-\bar{f}(x(t))$. Observe that $x(t)$ lies in a $(k-1)$-dimensional face of $D(t) \cap C$, where $k$ equals the dimension of the simplex having $x(t)$ in its interior. Figure 3 illustrates the stationary point interpretation for $n = 2$. Observe that $\bar{f}_2(x(t^*)) = 0$ and $\bar{f}_1(x(t^*)) > 0$, so that $x(t^*)$ solves (1.1) for $\bar{f}$. For the case that $a = 0$ and the $b_i$'s are plus infinite, this interpretation coincides with the one given in [ 3] when the starting point $v$ is chosen to be the zero point ($v = a$).

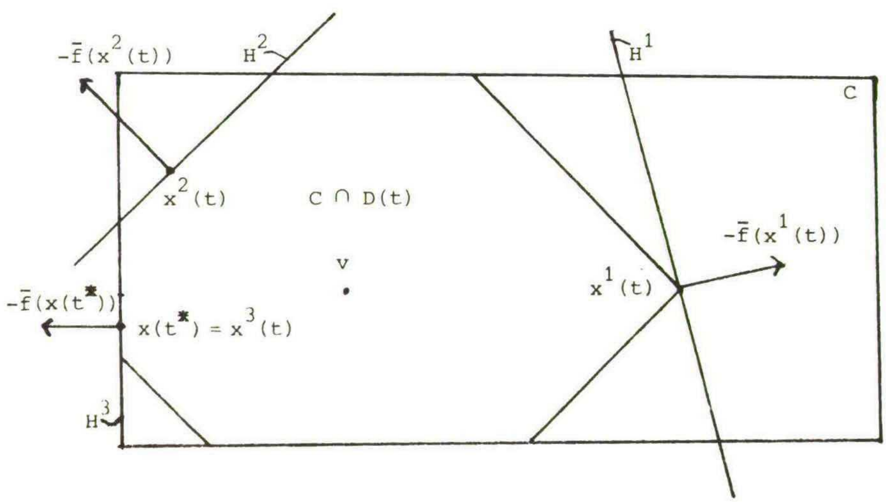


Figure 3. $C \cap D(t) \subset \bar{H}^i$ with $H^i$ the hyperplane through $x^i(t)$ with normal $\bar{f}(x^i(t))$. There are three stationary point solutions. The point $x(t^*)$ is a solution to problem (1.1) for $\bar{f}$, $(t > t^*)$; $x^1(t)$ lies in $A(1)$, $x^2(t)$ lies in $A(-1,2)$, $x^3(t)$ lies in $A(-2)$.

## 6. Applications.

Let us consider the computation of an equilibrium in an N-person game, each person having 2 pure strategies. Let $S^1 = \{x \in R^2_+ | x_1+x_2=1\}$ be the set of mixed strategies of each player and let S denote the strategy space of the game, i.e., $S = \Pi^N_{i=1} S^1$. The 2N-dimensional vector $(x^1,\dots,x^N)$ will denote an element of S such that $x^j \in S^1$, $j \in I_N$. For some $x \in S$, the marginal loss to player j if he plays his h-th pure strategy, h = 1,2, and the other players stick on strategy x is given by $m^j_h(x)$, with $m^j_h(x)$ a smooth continuous function from S to R. The expected loss $p^j(x)$ to player j if $x \in S$ is played is given by

$$p^j(x) = x^j_1\, m^j_1(x) + x^j_2 m^j_2(x) \qquad j = 1,\dots,N. \tag{6.1}$$

Definition 6.1. A point $\bar{x} \in S$ is an equilibrium strategy vector of the game if for each player j

$$m^j_h(\bar{x}) \geq p^j(\bar{x}) \qquad h = 1,2.$$

To prove the existence of such an equilibrium strategy, a function f from S into itself can be defined such that a fixed point of f is an equilibrium point and conversely. For the general N-person game, Van der Laan and Talman [6] gave a variable dimension restart algorithm to approximate a fixed point of f. However, in defining f, information is lost. Therefore they defined a nonlinear complementarity problem on S whose solution is an equilibrium point and they applied the algorithm on this problem.
This improved the computational results since less information is lost. However the problem is still far from smooth, whereas the algorithm cannot exploit the complementarity since on the boundary of S an artificial labelling is needed.

In this paper, however, we developed an algorithm to solve the nonlinear complementarity problem on a cubic region. For the case that each player has 2 pure strategies, the problem of finding an equilibrium can be transformed to an NLCP on a cubic region as follows.

Let $C^N$ be the N-dimensional unit cube, i.e. $C^N = \{x \in R^N | 0 \leq x_i \leq 1,$

$i = 1,\ldots,N\}$, and let $f : C^N \to R^N$ be defined by

$$f_j(x) = m_1^j(y(x)) - m_2^j(y(x)) \qquad j = 1,\ldots,N,$$

where $y(x)$ is the 2N-vector $(x_1, 1-x_1, x_2, 1-x_2,\ldots,x_N, 1-x_N)^T$ of S. Because of definition 6.1, the next theorem follows immediately.

Theorem 6.2. A point $\bar{x} \in C^N$ solves the NLCP on $C^N$ with respect to f, iff $\bar{y} \in S$ is an equilibrium strategy with $\bar{y} = y(\bar{x})$.

Now the NLCP on $C^n$ with respect to f can be solved by the algorithm of section 4, so that the complementarity can be exploited whereas the function f is still smooth.

As a second application we consider the constrained optimization problem

$$\min_x f(x) \qquad \text{s.t. } g_i(x) \geq 0,\ i = 1,\ldots,m, \text{ and } a \leq x \leq b, \qquad (6.2)$$

with f a convex and each $g_i$ a concave continuous differentiable function from $R^n$ to R.
To solve this problem, we consider for some fixed $u \geq 0$, the problem

$$\min h(x,u) = f(x) - \sum_{i=1}^{m} u_i g_i(x), \qquad (6.3)$$

subject to $a \leq x \leq b$. Let $x(u)$ solve (6.3). We assume that $x(u)$ is a continuous function of u. Now we define $z : R^m \to R^m$ by

$$z_i(u) = g_i(x(u)) \qquad i = 1,\ldots,m. \qquad (6.4)$$

Since both x and $g_i$, $i = 1,\ldots,m$, are continuous, it follows that z is continuous. Suppose that $u^*$ solves the NLCP on $R^m_+$ with respect to z, i.e. $z_i(u^*) \geq 0$ with equality when $u_i^* > 0$, $i = 1,\ldots,m$, then we will show that $x(u^*)$ solves (6.2).

Lemma 6.3. Let $u^*$ solve the NLCP with respect to z, then $x(u^*)$ minimizes $f(x)$ subject to $a \leq x \leq b$ and $g_i(x) \geq 0$, $i = 1,\ldots,m$.

Proof. Since $z_i(u^*) \geq 0$, we have from (6.4) $g_i(x(u^*)) \geq 0$, so that the point $x(u^*)$ is feasible. Moreover, $u_i^* z_i(u^*) = 0$, $i = 1,\ldots,m$. Therefore,

$$f(x(u^*)) = h(x(u^*),\ u^*) \leq h(x,u^*) = f(x) - \sum_{i=1}^{m} u_i^* g_i(x) \leq f(x)$$

for all feasible x.
Hence, $x(u^*)$ solves (6.2).

□

To solve the NLCP on $R_+^m$ with respect to z, we can apply the algorithm of section 4. At each function evaluation (6.3) must be solved subject to $a \leq x \leq b$. However, this problem is in fact an NLCP on $C^n = \{x \in R^n | a \leq x \leq b\}$ with respect to $\nabla_x h(x,u) = \nabla f(x) - \Sigma_{i=1}^m u_i \nabla g_i(x)$ and can also be solved with the algorithm of section 4. Recall that this algorithm gives an approximate solution to the problem. The accuracy of the approximation to the problem (6.3) should depend on the grid size of the triangulation of $R_+^m$ to solve the main problem (6.4). As a starting point of each subproblem (6.3), the approximation of the previous subproblem could serve in which case one run should be enough.

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